

# Finite Duration and Energy Effects in Lorentz-Violating Vacuum Cerenkov Radiation

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## Abstract

Vacuum Cerenkov radiation is possible in certain Lorentz-violating quantum field theories, when very energetic charges move faster than the phase speed of light. In the presence of a CPT-even, Lorentz-violating modification of the photon sector, the character of the Cerenkov process is controlled by the high-frequency behavior of the radiation spectrum. The development of the Cerenkov process can be markedly different, depending on whether the only limits on the emission of very energetic photons come from energy-momentum conservation or whether there are additional effects that cut off the spectrum at high frequencies. Moreover, since the high-frequency cutoff determines the total rate at which an emitting charge loses energy, it also controls all aspects of the emission that are related to the process's finite duration.

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# 1 Introduction

In roughly the last decade, there has been a great surge in interest in the possibility that Lorentz invariance may not be exact. If Lorentz violation were discovered experimentally, it would be a discovery of tremendous significance and would mean that there existed qualitatively new physics beyond general relativity and the standard model. However, despite many precision tests, there is thus far no evidence that relativity needs any modification.

Precision tests of Lorentz invariance are nothing new, but the field of Lorentz violation has changed substantially in recent years. For a long time, most tests of relativity were designed to search for *ad hoc* modifications of standard relativistic physics. That changed with the development of a systematic effective field theory approach. The standard model extension (SME) is an effective field theory that incorporates known physics and also the possibility of Lorentz violation [1, 2]. The violations enter through Lorentz noninvariant operators in the Lagrangian, parameterized by coefficient tensors with Lorentz indices. If Lorentz symmetry is broken spontaneously, these coefficients are the vacuum expectation values of tensor operators, selecting out preferred directions in spacetime. Some of these operators violate, in addition to Lorentz invariance, CPT invariance.

There are many ways that Lorentz symmetry can be violated in the SME. If nonrenormalizable terms are included in the Lagrangian, the number of coefficients characterizing the theory is infinite. The minimal SME contains only local, gauge invariant operators of dimension four or less that can be constructed out of standard model fields. The number of coefficients is still very large, but in most situations, only a relatively modest subset of them will affect a particular observable. For example, in many cases, only the Lorentz-violating coefficients for protons, neutrons, electrons, and photons come into play. These are the species we observe in low-energy physics experiments, and Lorentz violations in these sectors are fairly well bounded (whereas this is not so much the case for more exotic particles and fields).

Some effects which are absolutely forbidden in Lorentz invariant theories can occur readily in the SME. When the action is no longer invariant under Lorentz boosts, it is possible for different particles to have different maximum velocities. Specifically, it is possible for some particles to travel faster than the phase velocity of light (which is not necessarily energy independent). When charged particles move this fast, they must emit vacuum Cerenkov radiation. This kind of radiation is a unique signature of Lorentz violation, and it has already received a fair amount of attention [3, 4, 5, 6, 7, 8, 9]. However, vacuum Cerenkov radiation is by no means completely understood. In particular, it is not entirely clear what kind of role new physics entering at large energy scales will play in the process. We shall address that particular question in this paper. The expression for the total power radiated off by a superluminal charged particle is dominated by the ultraviolet end of the frequency spectrum. This means that anything dependent on this total power will depend on the spectrum's high-energy cutoff. The total emission rate determines all the properties of the Cerenkov process that are tied to its finite duration.

We shall therefore focus on how finite energy and finite duration effects play out in the vacuum Cerenkov process. The paper is organized as follows. In the remainder of the introduction, we shall consider the particular CPT-even model that will be discussed in the rest of the paper. We shall then examine the effects of the one cutoff for the emission spectrum that is guaranteed to exist—the cutoff due to energy-momentum conservation—in section 2. In section 3, we examine the competing effects of other possible cutoffs, including one whose existence is strongly suggested by naturalness considerations. Then we turn in section 4 to a study of diffraction in the Cerenkov process; this topic is interesting in itself and also draws together many of the results from earlier in the paper. We conclude in section 5 with a narrative describing how the vacuum Cerenkov process evolves over time and some additional remarks.

The least constrained operators in the photon sector of the minimal SME are part of the tensor  $k_F^{\mu\nu\rho\sigma}$  appearing in the electromagnetic Lagrange density

$$\mathcal{L}_F = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{4}k_F^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}. \quad (1)$$

There are nineteen independent parameters contained in the CPT-even  $k_F^{\mu\nu\rho\sigma}$ . Ten of them are associated with photon birefringence and have been very strongly constrained with cosmological measurements [10, 11]. (All the CPT-odd parameters are likewise very tightly bounded [12, 13].) The terms that do not lead to birefringence form a two-index traceless symmetric tensor  $\tilde{k}^{\mu\nu}$ , and if all the birefringent terms are zero,

$$k_F^{\mu\nu\rho\sigma} = \frac{1}{2} \left( g^{\mu\rho}\tilde{k}^{\nu\sigma} - g^{\mu\sigma}\tilde{k}^{\nu\rho} - g^{\nu\rho}\tilde{k}^{\mu\sigma} + g^{\nu\sigma}\tilde{k}^{\mu\rho} \right). \quad (2)$$

Most of the results in this paper can be generalized to cover the case of the most general  $k_F^{\mu\nu\rho\sigma}$ . We must simply split up the two polarizations, which propagate at different rates. Calculations of the Cerenkov spectrum in the presence of this birefringence is discussed in detail in [8]. However, we shall restrict our explicit calculations here to the  $\tilde{k}^{\mu\nu}$  only case. We shall work to leading order in  $\tilde{k}^{\mu\nu}$ , because Lorentz violation is supposed to be a small effect, and any higher order corrections must be miniscule.

For simplicity, we shall also not consider directly any modifications to the charged matter, which will generally affect the relationships between charged particles' momenta and velocities. Perhaps the simplest form of Lorentz violation for a fermion is given by

$$\mathcal{L}_\psi = \bar{\psi}[(\gamma^\mu + c^{\nu\mu}\gamma_\nu)i\partial_\mu - m]\psi. \quad (3)$$

Radiative corrections mix  $\tilde{k}^{\mu\nu}$  and the  $c^{\nu\mu}$  terms for charged species, so we do not expect the matter sector in the presence of  $\tilde{k}^{\mu\nu}$  to be truly conventional. However, for the purposes of determining the vacuum Cerenkov radiation, we may assume that the  $c^{\nu\mu}$  relevant to the moving charges vanishes, so long as we henceforth take the effective  $\tilde{k}^{\mu\nu}$  to be  $\tilde{k}_0^{\mu\nu} - 2c^{\nu\mu}$ , where  $\tilde{k}_0^{\mu\nu}$  is the true Lorentz-violating parameter appearing in the photon Lagrangian [14].

We shall be studying what happens to a charged particle moving in a direction  $\hat{v}$  with a speed  $v$  very close to 1. The phase speed at which light propagates in this direction is  $1 - \frac{1}{2} [\tilde{k}_{jk} \hat{v}_j \hat{v}_k + 2\tilde{k}_{0j} \hat{v}_j + \tilde{k}_{00}]$ . If  $v$  is greater than this, Cerenkov radiation will occur. Since the phase speed of the radiation can only deviate very slightly from 1, the Cerenkov cone will be very narrow. All the radiation is beamed into a narrow pencil of angles around  $\hat{v}$ , and so the direction dependence of the phase speed can be ignored. (However, if we did take into account the fact that the emitted photons do not travel in precisely the same direction  $\hat{v}$  as the charge, there would be higher order corrections that would deform the Cerenkov cone so that it would no longer be right angled or circular.)

Since no directions other than  $\hat{v}$  are involved in the Cerenkov process, the leading order effects are generally identical to what one would see if the charge were moving with a speed  $v$  in an isotropic medium with index of refraction

$$n = \left\{ 1 - \frac{1}{2} [\tilde{k}_{jk} \hat{v}_j \hat{v}_k + 2\tilde{k}_{0j} \hat{v}_j + \tilde{k}_{00}] \right\}^{-1}. \quad (4)$$

If  $n \leq 1$ , Cerenkov radiation is obviously impossible. If  $n > 1$ , there is radiation if the charge's energy exceeds the threshold energy

$$E_T = \frac{m}{\sqrt{\tilde{k}_{jk} \hat{v}_j \hat{v}_k + 2\tilde{k}_{0j} \hat{v}_j + \tilde{k}_{00}}}. \quad (5)$$

To make contact with the usual expressions describing Cerenkov radiation, which were derived for the case of radiation in a medium, we shall make frequent use of the effective refractive index  $n$ .

With  $n$  and  $v$  alone, it is already possible to calculate such quantities as the Cerenkov angle and the power spectrum for a steady state Cerenkov process. However, this does not capture all the relevant physics; other effects are also quite important. The steady state analysis neglects recoil corrections, which are related to the corpuscular nature of light. One obvious role that the backreaction on a radiating charge must play is as an ultraviolet regulator for the total power emitted. A charge cannot radiate away more energy than it possesses. With the backreaction taken into account, the radiation must cease after a finite time, and when the radiating track length is finite, diffraction can play an interesting role. Moreover, new physics that is important only at high energies might also come into play.

## 2 Recoil Corrections

Before we consider the impact of any new physics, we should look at how the well understood effects of energy and momentum conservation affect the Lorentz-violating Cerenkov process. Obviously, a charge cannot emit photons with arbitrarily high frequencies; the

energy is simply not available. The details of how recoil corrects the Cerenkov spectrum are comparatively simple, and the relevant calculations generally mirror those relevant to Cerenkov radiation in media. We can carry over the standard results using our prescription for  $n$  and making any additional approximations that are appropriate.

There is a well known result for the maximum frequency present in the Cerenkov spectrum emitted by a charge moving in a perfect, nondispersive dielectric [15]. This frequency is determined by energy-momentum conservation during the emission of a single photon. The maximum frequency for a charge with energy  $E$  and speed  $v$  is

$$\omega_m = 2E \frac{vn - 1}{n^2 - 1}, \quad (6)$$

where  $n$  is the index of refraction. We shall recast this expression in a more useful form. At ultrarelativistic energies, where the charge's energy is approximately  $E = m/\sqrt{2(1-v)}$ , so that

$$v = 1 - \frac{m^2}{2E^2}, \quad (7)$$

this reduces to

$$\omega_m = \frac{2E}{n+1} - \frac{m^2}{E} \frac{n}{n^2-1}. \quad (8)$$

In this regime, The threshold  $E_T$  for low-frequency Cerenkov emission is the energy at which  $v$  is equal to the speed of light  $n^{-1}$  in the medium. If  $n$  is close to 1, so  $1 - n^{-1} \approx n - 1 \approx m^2/2E_T^2$ , then the expression for  $\omega_m$  becomes

$$\omega_m = \frac{E^2 - E_T^2}{E}. \quad (9)$$

Importantly,  $\omega_m = (E - E_T) \left( \frac{E+E_T}{E} \right)$  is always greater than  $E - E_T$ . So whenever there is Cerenkov emission, there is a finite probability per unit time of emitting a photon near the upper end of the allowed frequency spectrum, so that the charge drops below the threshold energy, and further emission is impossible. It is the rate of emission of  $\omega > E - E_T$  photons that determines how long the Cerenkov process will last.

A more general version of the recoil analysis gives the change in the Cerenkov angle  $\theta_C$  due to recoil effects. The modified expression is [15]

$$\cos \theta_C = \frac{1}{vn} \left[ 1 + \frac{\omega}{2E} (n^2 - 1) \right], \quad (10)$$

and  $\omega_m$  is the frequency at which  $\theta_C$  shrinks to zero, cutting off the emission. To leading order in  $n - 1$ ,  $\theta_C$  is given by

$$\sin^2 \theta_C = 2 \left[ 1 - \frac{1}{v} + \left( 1 - \frac{\omega}{E} \right) \frac{n-1}{v} \right], \quad (11)$$

and at high energies,  $1 - v^{-1} \approx v - 1$  is given by (7).

As a lowest-order approximation for the emitted power that accounts for recoil corrections, we may simply use the recoil-corrected value of the Cerenkov angle in the power spectrum formula,  $P(\omega) = \frac{e^2}{4\pi} \sin^2 \theta_C \omega$ . This is essentially a phase space estimate, using a matrix element for the emission process that does not include recoil corrections but including the full effects of the recoil in the kinematics. In this approximation, and to leading order in  $n - 1$  (and hence leading order in the Lorentz violation), the rate of photon emission per unit frequency is

$$\Gamma(\omega) = \frac{P(\omega)}{\omega} = \frac{e^2 m^2}{4\pi} \left[ E_T^{-2} \left( 1 - \frac{\omega}{E} \right) - E^{-2} \right]. \quad (12)$$

If there is no cutoff other than that provided by the backreaction and energy-momentum conservation, the instantaneous rate of emission for all photons with energies greater than  $E - E_T$  is

$$\Gamma \equiv \int_{E-E_T}^{\omega_m} d\omega \Gamma(\omega) = \frac{e^2 m^2}{8\pi} \frac{(E - E_T)^2}{E^3}. \quad (13)$$

This is the instantaneous decay constant for the process in which the charge emits a single high-energy photon, drops below the Cerenkov threshold, and consequently stops emitting.

The rate  $\Gamma$  is small when the energy  $E$  is only slightly above the threshold, and it increases to a maximum value of  $\Gamma = \frac{e^2 m^2}{54\pi E_T}$  at  $E = 3E_T$ , so the probability per unit time of the Cerenkov process coming to a sudden halt never exceeds  $\sim 10^{-4} m \left| \tilde{k} \right|^{1/2}$ . At the highest energies, the rate behaves as  $\Gamma \approx \frac{e^2 m^2}{8\pi E}$ .

$\Gamma$  represents the rate for one kind of energy loss. Energy is also lost through the emission of lower-energy photons with  $\omega < E - E_T$ . The emission of one of these photons will not lower the energy to below  $E_T$ , and so the charge will continue to radiate afterwards. This makes it reasonable to approximate the energy losses from  $\omega < E - E_T$  photons as a continuous process, radiating power at a rate

$$P_{<} \equiv \int_0^{E-E_T} d\omega P(\omega) = \frac{e^2 m^2}{24\pi} \frac{(E - E_T)^3 (E + 3E_T)}{E^2 E_T^2}, \quad (14)$$

$P_{<}$  increases as  $E^2$  at large energies, when  $E \gg E_T$ . In this regime, the time scale for the charge to lose a substantial fraction of its energy by emitting lower-energy,  $\omega < E - E_T$ , photons is

$$\frac{E}{P_{<}} \approx \frac{24\pi E_T^2}{e^2 m^2 E}, \quad (15)$$

which is a much shorter time scale than  $\Gamma^{-1}$ . At high energies, the continuous emission of lower-energy photons is more important than the possibility of a single high-energy event that drops the particle below threshold. However, when  $E$  is only slightly greater

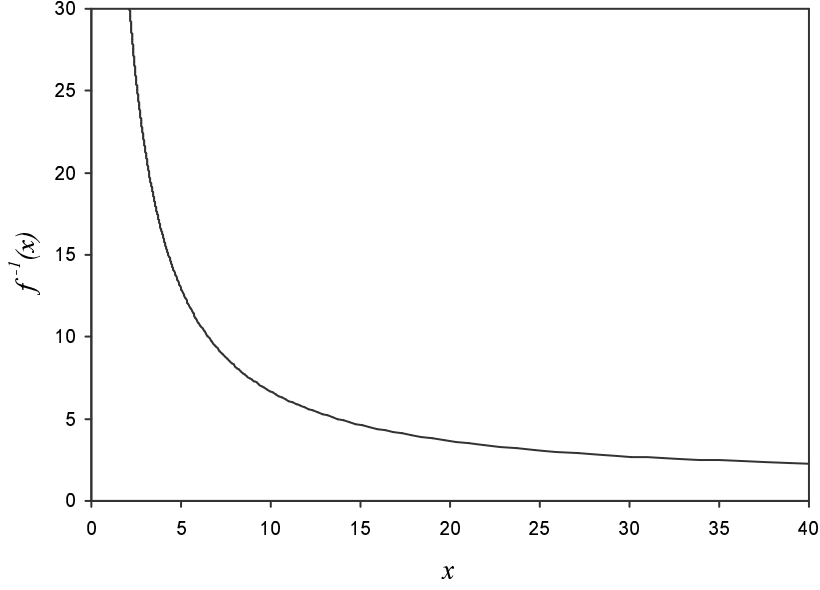


Figure 1: The function  $f^{-1}(x)$ , showing its characteristic behavior both at large and small  $x$ .

than  $E_T$ , the characteristic scales  $(E - E_T)/P_<$  and  $\Gamma^{-1}$  for the two types of losses are comparable.

We can combine  $\Gamma$  and  $P_<$  to find the time dependence of  $\Gamma$ . To do this, we approximate the energy loss coming from the lower-frequency radiation as deterministic, neglecting the decomposition of the emission into photons. We can then solve for the energy, given that no photon with an energy above  $E - E_T$  is emitted before a time  $t$ , by solving  $\dot{E} = -P_<$ . The solution is elementary—

$$E = E_T f^{-1} \left[ \frac{8e^2 m^2 t}{3\pi E_T} + f \left( \frac{E_0}{E_T} \right) \right], \quad (16)$$

where  $E_0$  is the charge's energy at  $t = 0$  and  $f^{-1}$  is the inverse of the function

$$f(x) = \frac{4(7x - 5)}{(x - 1)^2} + 9 \log \frac{x + 3}{x - 1}. \quad (17)$$

Combined with (13), this gives the time dependence of  $\Gamma$ . The function  $f^{-1}(x)$  is plotted in figure 1. As  $x \rightarrow 0$ ,  $f^{-1}(x) \approx 28/x$ , and this governs the behavior of the energy at small times and when  $E_0$  is large. For large values of  $x$ ,  $f^{-1}(x) \approx 1 + 2\sqrt{2/x}$ , indicating a more gradual rate of energy loss as the energy drops close to  $E_T$ .

### 3 Ultraviolet Cutoffs

Thus far, the only ultraviolet cutoff for the radiation that we have considered is  $\omega_m$ , whose existence is guaranteed by energy and momentum conservation. However, this is not necessarily the only cutoff that might affect the theory or even the most significant one. The most important cutoff will be whatever one lies the lowest in energy. In this section, we shall look at possibly relevant cutoffs, considering them separately to see how they compare.

The energy conservation cutoff is  $\omega_m$ . Just above threshold, when  $E - E_T \ll E_T$ , this cutoff may be quite small. Obviously, for sufficiently small  $E - E_T$ , this must be the most relevant cutoff. What happens at greater energies is less clear. Beyond the  $E - E_T \ll E_T$  regime, we have  $\omega_m \approx E$ , the cutoff growing linearly with the charge's energy scale. From (11), it is evident that the Cerenkov angle—which governs the emission rate—is little modified by the cutoff except for photon frequencies comparable to  $E$ . The scale  $E_T$ , when it is comparable to or lower than  $E$ , does not play any role in determining the frequency cutoff; nor does  $m$ .  $E$  provides the only scale involved. On dimensional grounds alone, the rate of energy loss must then be proportional to  $E^2$ , which indeed it is; the rate at which the charge loses energy smoothly is just  $P_<$ , which has the required energy dependence.

The electromagnetic sector, including  $\tilde{k}^{\mu\nu}$ , is invariant under dilation; it contains no preferred scale. However, in any theory with massive charged particles, the scale invariance is broken. There are Lorentz-violating operators in the charged fermion sector parameterized by coefficients  $c^{\nu\mu}$ , which mix with the  $\tilde{k}^{\mu\nu}$  operators under renormalization [16]. In a natural theory, the  $c^{\nu\mu}$  coefficients cannot be smaller than the  $\tilde{k}^{\mu\nu}$  [except possibly by a factor of  $\mathcal{O}(\alpha)$ ]. What is important about the existence of the  $c^{\nu\mu}$  terms is that they introduce another important scale into the theory beyond the fermion mass scale. The fermion sector will begin to have problems with stability or causality when particles reach momenta  $\sim m|c|^{-1/2}$  [17].

What happens at this scale can be understood as follows. The maximum achievable velocity (MAV) for a species of fermions depends on its  $c^{\nu\mu}$ . If the coefficients are such that the MAV in a given direction exceeds 1, then there are obviously causality problems. Particles with large momenta along the relevant direction in one frame will be able to travel superluminally, which means backwards in time as measured in a different observer frame. The stability problems occur if the MAV is less than 1. Then there are on-shell particle states with spacelike momenta, and in sufficiently boosted frames, these states will have negative energies, destabilizing the vacuum. The momentum scale at which either of these problems first becomes evident can readily be seen to be  $m|c|^{-1/2}$ . Some of the best bounds on electron Lorentz violation actually come from constraining the deviation of the electron MAV from 1, using, among other techniques, the observed absence of vacuum Cerenkov radiation in the spectra of energetic astrophysical sources [18, 19, 20, 21].

If new physics intervenes at the scale  $\Lambda_c \sim m|c|^{-1/2}$  to preserve some form of causality,



we expect the new interactions to deform the effective energy-momentum relation in such a way as to counteract the effects of  $c^{\nu\mu}$ . The simplest way to do this would be with higher-dimension, energy-dependent operators that keep the dispersion relations from going outside the null cone at high energies. This is not the only possibility, however. What is important is that it is natural (although one cannot say required) that there be new physics entering at energies  $\Lambda_{\tilde{k}} \sim m \left| \tilde{k} \right|^{-1/2}$  which will cut off the Cerenkov radiation, possibly by restoring the photon dispersion relation to its conventional  $\omega = \left| \vec{k} \right|$  form at higher momenta.

The mass scale  $m$  appearing in  $\Lambda_{\tilde{k}}$  represents the mass of the lightest charged particle (physically, the electron). In principle, each species has its own coefficients  $c^{\nu\mu}$  (which actually do not need to be diagonal in flavor space), but naturalness dictates that all the  $c^{\nu\mu}$  should be comparable in size. The scales at which causality problems occur are not the same for the various species in the case, and if new physics is to rescue this property of the theory, it must become important at the smallest scale where troubles might be seen.

The momentum scale  $\Lambda_{\tilde{k}}$  is of the same order as the threshold energy  $E_T$  (assuming the radiating particle is a representative of the lightest species; if it is not, then  $\Lambda_{\tilde{k}}$  is smaller than the typical  $E_T$ ). Assuming that there is indeed a cutoff in the Cerenkov spectrum at a frequency  $\Lambda_{\tilde{k}}$ , this will be a lower cutoff than  $\omega_m$ , except in the limited range of energies  $E - E_T \ll E_T$ . At high energies, the continuous energy loss is no longer given by  $P_<$ , but instead by

$$P = \frac{e^2 m^2}{8\pi} \left( \frac{\theta_C^2 \Lambda_{\tilde{k}}^2}{m^2} \right), \quad (18)$$

where  $\theta_C$  means the zero-frequency value of the Cerenkov angle. Most of the energy is emitted in the highest allowed frequency modes with  $\omega \sim \Lambda_{\tilde{k}}$ . There is no guarantee that any new physics should not be Lorentz violating itself, and the value of  $\Lambda_{\tilde{k}}$  relevant for this calculation may well depend on the direction of the charge's motion, just as  $E_T$  depends on  $\hat{v}$ . So the expression in parenthesis in (18) may depend on orientation, but its order of magnitude is fixed. It is dimensionless and  $\mathcal{O}(1)$ , meaning that the charge radiates at a constant rate, which is independent not only of the energy  $E$  but also of the magnitude of the Lorentz violation. The smallness of the Lorentz violation is precisely compensated for by the largeness of the scale at which the cutoff occurs.

It is of course entirely possible that the new physics enters at a scale  $\Lambda$  other than  $\Lambda_{\tilde{k}}$ , though based on naturalness, we would expect this scale not to be larger than  $\Lambda_{\tilde{k}}$ . If the true cutoff scale is  $\Lambda$ , then (18) need only be modified by the substitution  $\Lambda_{\tilde{k}} \rightarrow \Lambda$ . In fact, using this formula with  $\Lambda = \sqrt{\frac{2}{3}} E$  reproduces the high-energy form of  $P_<$ , consistent with our earlier interpretation of  $\omega_m$  as simply introducing a cutoff at the scale  $E$ ; however, this is a cutoff that depends on the energy, and hence the power emitted is time dependent.

With a fixed cutoff, independent of  $E$ , the  $\omega > E - E_T$  decay rate  $\Gamma$  will also be

modified. With a sharp cutoff at a lower frequency, there is simply no emission of photons this energetic. Until the energy falls low enough that  $E - E_T < \Lambda$ ,  $\Gamma$  is zero. A sharp cutoff in frequency is probably unrealistic, so the rate  $\Gamma$  will probably always remain nonzero. However, the emission of any photon with an energy greater than  $\Lambda$  should be strongly suppressed. The emission of the most energetic photons, which already represents a slower form of energy loss than the continuous lower-energy emission even in the absence of  $\Lambda$ , becomes essentially completely negligible as a loss mechanism until the charge's energy has fallen low enough so that  $E - E_T$  is comparable to  $\Lambda$ . For  $\Lambda = \Lambda_{\tilde{k}}$ , this occurs when  $E \sim E_T$ .

## 4 Diffraction

Having examined how various ultraviolet cutoffs might come into play, we shall now discuss a topic that is interesting in its own right but also serves to demonstrate the complexity with which the multiple scales in the Lorentz-violating Cerenkov process—particularly the ultraviolet cutoffs—interact. Diffraction of the Cerenkov radiation turns out to depend crucially on how the high-energy photon spectrum is cut off. Because it is losing energy, a moving charge cannot continue emitting Cerenkov radiation forever. The process must have a limited duration, and the fact that the resulting track length is finite causes the lower-energy Cerenkov radiation to diffract. In this way, the recoil from the most energetic photons indirectly affects the angular distribution of the least energetic ones. The classical expression for the diffraction width, for the Cerenkov radiation emitted by a charge that moves superluminally for a distance  $L$  without losing significant energy, is  $\Delta\theta \sim \lambda/(L \sin \theta_C)$ .  $L \sin \theta_C$  is the distance the charge moves perpendicular to the direction of the photon emission, assuming that  $\theta_C$  is not so small as to be comparable to  $\Delta\theta$ .

However, in the case of interest here,  $\theta_C$  is always small. Normally,  $\theta_C$  gives the width of the cone into which the radiation is emitted. However, if the diffraction width is comparable to or larger than  $\theta_C$ , it is  $\Delta\theta$  that sets the size of this cone. Of course, the two radiation cones are structurally very different. In the idealized case of no dispersion and no recoil, all the photons are emitted on the surface of the cone. When the cone width is set by diffraction, the photons are smeared out over the full characteristic angular width  $\Delta\theta$ . The distance that the charge moves perpendicular to the direction of photon emission is always  $\sim L \sin \theta_w$ , where  $\theta_w$  is the width of the cone, so in the regime where  $\Delta\theta$  is dominant,  $\Delta\theta \sim \lambda/(L \sin \Delta\theta)$ , or  $\Delta\theta \sim \sqrt{\lambda/L}$ . (For extremely low frequency photons, whose wavelengths are not small compared to  $L$ , there is another regime. There the diffractive effects spread the radiation out over a broad range of angles  $\Delta\theta \sim 1$ .)

Obviously, the changeover between the  $\theta_w \approx \theta_C$  regime and the  $\theta_w \sim \Delta\theta$  regime occurs at the frequency for which  $\theta_C(\omega) \sim 1/\sqrt{\omega L}$ . If the frequency in question is low enough that recoil corrections can be ignored and  $\theta_C^2 \approx 2(v + n - 2)$ , the crossover occurs at

$\omega \sim 1/[(v+n-2)L]$ . However, determining the correct value of  $L$  is tricky. Two photons with the same frequency but emitted at two different points can only interfere in the far field if the Cerenkov angle  $\theta_C$  does not change appreciably in the time between the two emissions. By an appreciable change, we mean one that is larger than the instantaneous angular width into which photons of a fixed frequency are emitted; but this last width is just  $\Delta\theta$ , which is comparable to  $\theta_C$  in the crossover situation we are presently considering. Therefore,  $L$  must be the distance the charge needs to travel so that its energy loss will narrow the Cerenkov cone by an  $\mathcal{O}(1)$  factor. For the low-frequency value of  $\theta_C$  to be cut in half, the charge's energy must fall from its initial value of  $E_0$  to

$$E_1 = \frac{E_0 E_T}{\sqrt{\frac{3}{4}E_0^2 + \frac{1}{4}E_T^2}} \sim E_T. \quad (19)$$

In general, for  $\theta_C$  to change substantially, the energy must fall to a value comparable to  $E_T$ . The reason from this behavior is fairly clear. The velocity, upon which  $\theta_C$  depends, goes to 1 at high energies, depending less and less on  $E$  as  $E$  increases; in this regime,  $\theta_C$  is completely determined by  $n$ . Only when  $n-1$  and  $1-v$  are comparable (that is, near the Cerenkov threshold) does  $\theta_C$  depend significantly on  $E$ .

So  $L$  is determined by the rate at which the charge loses energy. If the initial energy is large, the charge will lose most of its energy as it traverses the distance  $L$ . With a fixed cutoff  $\Lambda$ , this requires a time  $\tau_\Lambda \approx \frac{8\pi(E_0-E_T)}{e^2\theta_C^2\Lambda^2}$ . If the only cutoff is provided at  $\omega_m$  by energy-momentum conservation, the time required for the energy to fall this low is determined by the behavior of  $f^{-1}(x)$ . In this case, an energetic charge loses energy very quickly initially, but the loss rate decreases with declining  $E$ . The charge reaches an energy not too far above  $E_T$  and decays relatively slowly after that, according to  $E = E_T + \sqrt{3\pi E_T^3/e^2 m^2 t}$ . The characteristic time  $\tau_m \approx \frac{3\pi E_T}{e^2 m^2}$  matches our earlier estimate of  $E/P_<$ , if we take  $E \approx 8E_T$  to be the characteristic energy  $E$  of the latter estimate, and this value accords nicely with our qualitative arguments. The time  $\tau_m$  is also comparable to  $\Gamma^{-1}$  in the  $E \sim E_T$  regime, so  $\tau_m$  represents the only relevant time scale arising from the cutoff  $\omega_m$ .

In any prolonged Cerenkov energy loss process (that is, one in which none of the  $\omega > E - E_T$  photons which would bring all further emission to a sudden halt are emitted), the  $\omega_m$  cutoff will eventually become the dominant one, simply because the energy must reach a point where  $E - E_T \ll \Lambda$ . So if there is a cutoff  $\Lambda$  distinct from  $\omega_m$ , the length  $L$  is determined by the distance the charge travels in the longer of the two times  $\tau_\Lambda$  and  $\tau_m$ . Remembering that  $\theta_C^2 \sim \frac{m^2}{E_T^2}$ , we see that  $\tau_\Lambda$  and  $\tau_m$  are comparable if  $\Lambda^2 \sim E_0 E_T$ . For a cutoff  $\Lambda_{\tilde{k}}$  at the natural scale  $E_T$ ,  $\tau_\Lambda$  is larger unless the initial energy  $E_0$  is also comparable to  $E_T$ . A smaller  $\Lambda$  only makes  $\tau_\Lambda$  larger, so in all cases with  $\Lambda \lesssim E_T$ , the time  $\tau_\Lambda$  predominates. More generally,

$$L \sim \frac{E_T}{e^2 m^2} \max \left( 1, \frac{E_0 E_T}{\Lambda^2} \right). \quad (20)$$

Assuming that there is indeed a cutoff  $\Lambda \lesssim E_T$ , this means that for frequencies  $\omega \lesssim \frac{e^2 \Lambda^2}{E_0}$ , the diffraction width  $\Delta\theta \sim \sqrt{\lambda/L}$  is larger than the Cerenkov angle  $\theta_C$ . The location of the change in regimes is tied critically to the value of the cutoff. Diffraction originates from the fact that radiation emitted in a finite region of space cannot be in a pure momentum eigenstate. Some photons must be emitted along directions other than those specified by energy-momentum conservation considerations. This suggests the possibility that diffraction might affect the rates of energy and momentum loss by the moving charge, the potential complications becoming most serious for very short track lengths. However, it turns out that this is not actually a problem in this situation. Since  $\Lambda < E_0$ , the frequencies for which diffraction is important are all small compared with the cutoff, and photons emitted away from the angle  $\theta_C$  do not significantly affect the rate of energy-momentum loss. This was a necessary consistency check for all our earlier calculations.

## 5 Conclusion

Vacuum Cerenkov radiation is a very special feature of Lorentz-violating theories. In this paper, we have described some further properties of the Cerenkov process in the presence of a CPT-even form of Lorentz violation. The emission rate and other physically significant quantities are controlled by high frequency cutoffs—and not necessarily in obvious ways.

Understanding the backreaction of the emitted radiation on the charge was crucial, since this is what determines the time evolution of the Cerenkov process. The progress of the physical process is actually rather subtle, and it depends on the ultraviolet structure of the theory. Photon emissions can be divided into two very different types, based on the frequencies of the photons involved. Photons with frequencies below  $E - E_T$  are emitted more or less continuously, but for more energetic photons, the quantal nature of the emission is of paramount importance. As soon as one of these extremely energetic photons is emitted, the charge's energy drops below the Cerenkov threshold, and the emission process abruptly terminates. Well above threshold, the total power emitted in the lower-energy modes is proportional to the square of the ultraviolet cutoff, while the rate at which  $\omega > E - E_T$  photons are emitted is increasingly suppressed at higher energies.

If there are no modifications of the photon sector other than the  $\tilde{k}^{\mu\nu}$ , the high-energy cutoff for the photon spectrum arises from energy conservation. The total emission rate for the lower-energy photons is approximately  $P_<$ , proportional to  $E^2$ . The decay rate  $\Gamma$  describing the higher-energy part of the spectrum is suppressed at high energies by  $E^{-1}$ . The charge will lose energy very quickly to start with, and it is unlikely to decay discontinuously before the energy has dropped to the scale  $E \sim E_T$ . Once it reaches that regime, the energy loss rates for the low and high frequency parts of the spectrum become

comparable. Most of the time, the particle will lose an  $\mathcal{O}(1)$  fraction of its remaining energy above threshold, then terminate the process with a single photon that disperses all the rest of the energy.

If there is another cutoff  $\Lambda$  dictated by new physics, the situation is different. Naturalness of the quantum corrections to this theory suggest that  $\Lambda$  should probably be no larger than  $\Lambda_{\tilde{k}} \sim E_T$ . However, as long as  $\Lambda < E - E_T$ ,  $\Lambda$  is the ultraviolet cutoff that controls the rate of energy loss. In this regime, the loss rate is independent of  $E$ , proportional instead to  $\Lambda^2$ . Moreover, if  $\Lambda \sim \Lambda_{\tilde{k}}$ , the energy loss rate is in fact independent of the scale of the Lorentz violation coefficients  $\tilde{k}^{\mu\nu}$  and is simply  $P \sim \frac{e^2 m^2}{8\pi}$ . In the regime where  $\Lambda$  is the predominant cutoff, emission of  $\omega > E - E_T$  photons is all but impossible, since these frequencies are above the cutoff scale; the discontinuous component of the energy loss is, if not completely vanishing, strongly suppressed. A charge beginning with a very large energy will radiate at a constant rate until it leaves the  $\Lambda$ -dominated regime. Once  $E - E_T < \Lambda$ , the process is cut off primarily by energy conservation effects, and the last phase of the process resembles what would be seen if  $\Lambda$  were not present.

The total track length depends on the cutoff, as given by (20). Although this expression for  $L$  was not explicitly derived as the total track length, it does represent the scale of that quantity. The total track length is determined by how long it takes for the charge to emit just one  $\omega > E - E_T$  photon. At high energies, the rate for such emission is very small; the charge must lose energy until  $E \sim E_T$  before the rate becomes appreciable. After that, the time scale required for such a decay is roughly  $\tau_m$ , so the total time (and hence total track length) is again set by the maximum of  $\tau_\Lambda$  and  $\tau_m$ .

The strong dependences on how the spectrum is cut off at high frequencies derive from the fact that, when all effects that might lead to a cutoff are neglected, the power spectrum grows rapidly at high frequencies. Once a cutoff is included, it sets the overall rate of energy loss, which determines how the process evolves. When there are several competing effects that all could potentially cut off the emission, whichever cutoff is smallest at a given time predominates. At high enough energies, the energy-momentum cutoff  $\omega_m \approx E$  will be greater than any  $E$ -independent cutoff. So only if there is no other fixed cutoff will  $\omega_m$  control the emission from the most energetic charges. That there should be no other energy-independent  $\Lambda$  is disfavored by naturalness and causality requirements, which suggest that new physics counteracting the effects of  $\tilde{k}^{\mu\nu}$  should enter at a scale  $\Lambda_{\tilde{k}}$  or lower.

The other components of  $k_F^{\mu\nu\rho\sigma}$  besides those contained in  $\tilde{k}^{\mu\nu}$  are not mixed with any renormalizable coefficients in the charged matter sector, so naturalness does not dictate any scale at which their effects are likely to be modified. If all the components of  $k_F^{\mu\nu\rho\sigma}$  are of the same order of magnitude, we might expect them all to be replaced by new physics at the same scale  $\Lambda_{\tilde{k}} \sim m |k_F|^{-1/2}$ ; however, this is by no means guaranteed. Moreover, if all the components of  $k_F^{\mu\nu\rho\sigma}$  are comparable, then the physical  $\tilde{k}^{\mu\nu}$  are constrained by the experimental bounds on the other components, which can be measured much more accurately because they lead to photon birefringence.

The  $k_F^{\mu\nu\rho\sigma}$  coefficients are unique in the photon sector of the SME, in that they are gauge invariant and dimensionless. Other forms of Lorentz violation are parameterized by dimensional constants, and these can introduce natural cutoff scales on their own. The cutoff dependences may not be so critical as they are in the case of the  $k_F^{\mu\nu\rho\sigma}$  terms, but the Cerenkov processes in the presence of these other forms of Lorentz violation (including nonrenormalizable forms) are still quite interesting, and more work is needed to understand them completely.

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## References

- [1] D. Colladay, V. A. Kostelecký, Phys. Rev. D **55**, 6760 (1997).
- [2] D. Colladay, V. A. Kostelecký, Phys. Rev. D **58**, 116002 (1998).
- [3] R. Lehnert, R. Potting, Phys. Rev. Lett. **93**, 110402 (2004).
- [4] R. Lehnert, R. Potting, Phys. Rev. D **70**, 125010 (2004).
- [5] C. Kaufhold, F. R. Klinkhamer, Nucl. Phys. B **734**, 1 (2006).
- [6] T. Jacobson, S. Liberati, D. Mattingly, Annals Phys. **321**, 150 (2006).
- [7] B. Altschul, Phys. Rev. Lett. **98**, 041603 (2007).
- [8] B. Altschul, Phys. Rev. D **75**, 105003 (2007).
- [9] C. Kaufhold, F. R. Klinkhamer, Phys. Rev. D **76**, 025024 (2007).
- [10] V. A. Kostelecký, M. Mewes, Phys. Rev. Lett. **87**, 251304 (2001).
- [11] V. A. Kostelecký, M. Mewes, Phys. Rev. Lett. **97**, 140401 (2006).
- [12] S. M. Carroll, G. B. Field, R. Jackiw, Phys. Rev. D **41**, 1231 (1990).
- [13] S. M. Carroll, G. B. Field, Phys. Rev. Lett. **79**, 2394 (1997).
- [14] Q. G. Bailey, V. A. Kostelecký, Phys. Rev. D **70**, 076006 (2004).
- [15] J. V. Jelley, *Cerenkov Radiation and its Applications* (Pergamon, New York, 1958), p. 28.

- [16] V. A. Kostelecký, C. D. Lane, A. G. M. Pickering, Phys. Rev. D **65**, 056006 (2002).
- [17] V. A. Kostelecký, R. Lehnert, Phys. Rev. D **63**, 065008 (2001).
- [18] T. Jacobson, S. Liberati, D. Mattingly, Nature **424**, 1019 (2003).
- [19] B. Altschul, Phys. Rev. Lett. **96**, 201101 (2006).
- [20] B. Altschul, Phys. Rev. D **74**, 083003 (2006).
- [21] B. Altschul, Phys. Rev. D **75**, 041301 (R) (2007).